

**COMMENTS ON: “OPERATOR K -THEORY FOR THE GROUP
 $SU(n, 1)$ ” BY P. JULG AND G. KASPAROV.**

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ABSTRACT. In this note we point out and fill a gap in the proof by Julg-Kasparov [JK] of the Baum-Connes conjecture with coefficients for discrete subgroups of $SU(n, 1)$. The issue at stake is the proof that the complex powers of the contact Laplacian are element of the Heisenberg calculus. In particular, we explain why we cannot implement into the setting of the Heisenberg calculus the classical Seeley’s approach to complex powers.

The aim of this note is to point out and fill a gap in the proof by Julg-Kasparov [JK] of the Baum-Connes conjecture with coefficients for discrete subgroups of the complex Lorentz group $G = SU(n, 1)$. The issue is with the proof by Julg and Kasparov that the complex powers of the contact Laplacian are pseudodifferential operators in the Heisenberg calculus of Beals-Greiner [BG] and Taylor [Ta].

To prove this Julg and Kasparov tried to carry out in the Heisenberg setting the classical approach of Seeley [Se] to complex powers of elliptic operators. We point a gap in their argument and we further show that we cannot implement Seeley’s approach to complex powers into the setting of the Heisenberg calculus. Nevertheless, the result about the complex powers of the contact Laplacian can be proved by using the results of [Po2]. This allows us to fill the gap in [JK].

The note is organized as follows. In Section 1 we point out explain a gap in the proof of Julg-Kasparov. In Section 2 we give a brief review of the Heisenberg calculus. In Section 3 we fill the gap and in Section 4 we explain why we actually can’t implement Seeley’s approach in the setting of the Heisenberg calculus.

1. A GAP IN JULG-KASPAROV’S PROOF

The proof by Julg and Kasparov in [JK] of the Baum-Connes conjecture with coefficients for discrete subgroups of $SU(n, 1)$ can be briefly summarized as follows.

First, the proof can be reduced to showing that Kasparov’s element γ_G is equal to 1 in the representation ring $R(G)$, that is, if K is the maximal compact group of G the restriction map $R(G) \rightarrow R(K)$ is an isomorphism.

Second, the symmetric space G/K is a complex hyperbolic space and under the Siegel map it is biholomorphic to the unit ball $B^{2n} \subset \mathbb{C}^n$ and its visual boundary is CR diffeomorphic to the unit sphere S^{2n-1} equipped its standard CR structure. Julg and Kasparov further showed that $R(K)$ can be geometrically realized as $KK_G(C(\overline{B}^{2n}), \mathbb{C})$, where $C(\overline{B}^{2n})$ denotes the C^* -algebra of continuous functions on the closed unit ball \overline{B}^{2n} and KK_G is the equivariant KK functor of Kasparov. They then built a Fredholm module representing an element δ in $KK_G(C^0(\overline{B}^{2n}), \mathbb{C})$ which is mapped to γ in $KK_G(\mathbb{C}, \mathbb{C}) = R(G)$ under the morphism induced by the map $\overline{B}^{2n} \rightarrow \{\text{pt}\}$.

The construction of the element δ in $KK_G(C(\overline{B}^{2n}), \mathbb{C})$ involves in a crucial manner the contact complex of Rumin [Ru] on the unit sphere S^{2n-1} endowed with its standard contact structure. In the contact setting the main geometric operators are not elliptic and the relevant pseudodifferential calculus to deal with them is the Heisenberg calculus of Beals-Greiner [BG] and Taylor [Ta]. Then for constructing the Fredholm module representing δ Julg and Kasparov had to prove that the complex powers of the contact Laplacian are pseudodifferential operators in the Heisenberg calculus (see [JK, Thm. 5.27]).

In [Se] Seeley settled a general procedure for constructing complex powers of elliptic operators as pseudodifferential operators. Its approach relied on constructing asymptotic resolvents in a suitable class of classical Ψ DO's calculus with parameter. Accordingly, Julg and Kasparov tried to construct an asymptotic resolvent for the contact Laplacian in a class of Heisenberg Ψ DO's with parameter λ in any given angular sector $\Lambda \subset \mathbb{C} \setminus (0, \infty)$ (see [JK, Thm. 5.25]). In particular, in their construction the symbol in a local chart of the asymptotic resolvent is never defined for $\lambda = 0$ and its homogeneous components have meromorphic singularities near $\lambda = 0$.

Now, in order to carry out Seeley's approach for the contact Laplacian we have to be able to integrate the symbol of the asymptotic resolvent with respect to the parameter λ over a contour Γ crossing the value $\lambda = 0$. This becomes troublesome when in the proof of [JK, Thm. 5.27] the authors claim that by their Theorem 5.25 the resolvent of the contact Laplacian belongs to a class of Heisenberg Ψ DO's with parameter in a set containing Γ . In particular, their Theorem 5.25 does not allow them to integrate over Γ the homogeneous components of symbol of the asymptotic resolvent. This shows that the proof of their Theorem 5.27 is not complete.

2. HEISENBERG CALCULUS

Let M^{2n-1} be a compact orientable contact manifold with contact hyperplane $H = \ker \theta$, where θ is a contact form, i.e., $d\theta|_H$ is non-degenerate. The contact condition implies that there is a non-degenerate Levi form $\mathcal{L} : H \times H \rightarrow TM/H$ such that, for any $x \in M$ and any sections X' and Y' of H we have $\mathcal{L}_x(X'(x), Y'(x)) = [X', Y'](x) \bmod H_x$, i.e., the value of $[X', Y'](x)$ modulo H_x only depends on the values at x of X' and Y' and not on their higher order jets. This allows us to define a tangent Lie group bundle GM as the bundle $(TM/H) \oplus H$ equipped with the dilations and group law such that

$$(1) \quad t.(X_0 + X') = t^2 X_0 + tX', \quad t \in \mathbb{R},$$

$$(2) \quad (X_0 + X').(Y_0 + Y') = X_0 + Y_0 + \frac{1}{2}\mathcal{L}(X', Y') + X' + Y',$$

for sections X_0 and Y_0 of TM/H and sections X' and Y' of H . Furthermore, since H is a contact hyperplane GM is in fact a fiber bundle of Lie groups with typical fiber the $(2n+1)$ -dimensional Heisenberg group \mathbb{H}^{2n-1} .

The Heisenberg calculus of Beals-Greiner [BG] and Taylor [Ta] is suitable pseudodifferential calculus to deal with hypoelliptic operators on contact manifolds. Its idea, which goes back to Eli Stein, is to construct a class of pseudodifferential operators, called Ψ_H DO's, whose calculus is modelled on that of convolutions operators on the Heisenberg group.

Locally $\Psi_H\text{DO}$'s can be described as follows. Let $U \subset \mathbb{R}^{2n-1}$ be a local chart with a H -frame X_0, X_1, \dots, X_{2n} of TU , i.e., a frame such that X_1, \dots, X_{2n} span H . In the sequel such a chart will be called a Heisenberg chart. In addition, we endow \mathbb{R}^{2n-1} with the pseudo-norm $\|\xi\| = (\xi_0^2 + \xi_1^4 + \dots + \xi_{2n}^4)^{\frac{1}{4}}$ and the dilations $t.\xi = (t^2\xi_0, t\xi_1, \dots, t\xi_{2n})$, $t \in \mathbb{R}$.

A Heisenberg symbol of order m , $m \in \mathbb{C}$, is a function $p \in C^\infty(U \times \mathbb{R}^{2n-1})$ admitting an asymptotic expansion $p \sim \sum_{j \geq 0} p_{m-j}$ with symbols $p_{m-j} \in C^\infty(U \times (\mathbb{R}^{2n-1} \setminus \{0\}))$ such that $p_{m-j}(x, t.\xi) = t^{m-j} p_{m-j}(x, \xi)$ for any $t > 0$. Here the sign \sim means that, for any compact $K \subset U$ and any integer N , we have estimates,

$$(3) \quad |\partial_x^\alpha \partial_\xi^\beta (p - \sum_{j < N} p_{m-j})(x, \xi)| \leq C_{KN\alpha\beta} \|\xi\|^{\Re m - N - \langle \beta \rangle}, \quad x \in K, \|\xi\| \geq 1,$$

where we have let $\langle \beta \rangle = 2\beta_0 + \beta_1 + \dots + \beta_{2n}$.

Let $\frac{1}{i}X_j = \sum_k a_{jk}(x) \partial_{x_k}$ and set $a(x) = (a_{jk}(x))$. A $\Psi_H\text{DO}$ of order m on U is a continuous operator $P : C_c^\infty(U) \rightarrow C^\infty(U)$ of the form,

$$(4) \quad Pu(x) = (2\pi)^{-(2n+1)} \int e^{i\langle x, \xi \rangle} p(x, a(x)\xi) \hat{u}(\xi) d\xi + Ru(x),$$

where $p(x, \xi)$ is a Heisenberg symbol of order m and R is a smoothing operator. The class of $\Psi_H\text{DO}$'s of order m is invariant under changes of Heisenberg charts and so we can define $\Psi_H\text{DO}$'s of order m on M acting on sections of a vector bundle \mathcal{E} over M . We let $\Psi_H(M, \mathcal{E})$ denote the class of such operators.

Let $\mathfrak{g}^*M = (TM/H)^* \oplus H^*$ be the linear dual of the Lie algebra bundle of GM . If P is an operator in $\Psi_H^m(M, \mathcal{E})$ then its principal symbol can be invariantly defined as an element $\sigma_m(P)(x, \xi)$ of the space $S_m(\mathfrak{g}^*M, \mathcal{E})$ of sections $p \in S_m(\mathfrak{g}^*M \setminus \{0\}, \text{End } \mathcal{E})$ such that $p(x, t.\xi) = t^m p(x, \xi)$ for any $t > 0$ (here \mathcal{E} is seen as a vector bundle over \mathfrak{g}^*M using the canonical submersion $\mathfrak{g}^*M \rightarrow M$).

Let $a \in M$ and let $S_m(\mathfrak{g}_a^*M, \mathcal{E}_a)$ be the space of functions $p \in C^\infty(\mathfrak{g}_a^*M \setminus \{0\}, \mathcal{E}_a)$ which are homogeneous of degree m . Then under the Fourier transform the convolution of distributions on G_aM defines a bilinear product $*^a$ from $S_{m_1}(\mathfrak{g}_a^*M, \mathcal{E}_a) \times S_{m_2}(\mathfrak{g}_a^*M, \mathcal{E}_a)$ to $S_{m_1+m_2}(\mathfrak{g}_a^*M, \mathcal{E}_a)$. This product depends smoothly on a , so it gives rise to the product,

$$(5) \quad * : S_{m_1}(\mathfrak{g}^*M, \mathcal{E}) \times S_{m_2}(\mathfrak{g}^*M, \mathcal{E}) \rightarrow S_{m_1+m_2}(\mathfrak{g}^*M, \mathcal{E}),$$

such that $p_{m_1} * p_{m_2}(a, \xi) = (p_{m_1}(a, \cdot) *^a p_{m_2}(a, \cdot))(\xi) \forall p_{m_j} \in S_{m_j}(\mathfrak{g}^*M, \mathcal{E})$. This product corresponds to the product of $\Psi_H\text{DO}$'s at the level of principal symbols, i.e., we have $\sigma_{m_1+m_2}(P_1 P_2) = \sigma_{m_1}(P_1) * \sigma_{m_2}(P_2) \forall P_j \in \Psi_H^{m_j}(M, \mathcal{E})$.

In fact, if in a given local trivializing Heisenberg chart with H -frame X_0, \dots, X_{2n-1} the operators $P \in \Psi_H^m(M, \mathcal{E})$ and $Q \in \Psi_H^{m'}(M, \mathcal{E})$ have symbols $p \sim \sum p_{m-j}$ and $q \sim \sum q_{m'-j}$ then PQ has symbol $r \sim \sum r_{m+m'-j}$, with

$$(6) \quad r_{m+m'-j} = \sum_{k+l \leq j} \sum_{\alpha, \beta, \gamma, \delta}^{(j-k-l)} h_{\alpha\beta\gamma\delta}(D_\xi^\delta p_{m-k}) * (\xi^\gamma \partial_x^\alpha \partial_\xi^\beta q_{m'-l}),$$

where $\sum_{\alpha\beta\gamma\delta}^{(k)}$ denotes the sum over all the indices such that $|\alpha| + |\beta| \leq \langle \beta \rangle - \langle \gamma \rangle + \langle \delta \rangle = k$ and $|\beta| = |\gamma|$, and the functions $h_{\alpha\beta\gamma\delta}(x)$'s are polynomials in the derivatives of the coefficients of the vector fields X_0, \dots, X_{2n-1} . It follows from this that we can

construct a parametrix of P in $\Psi_H^{-m}(M, \mathcal{E})$ if, and only if, its principal symbol $\sigma_m(P)$ is invertible with respect to the product $*$.

As $G_a M$ is not Abelian the product $*^a$ is not anymore the pointwise product of functions. Therefore, if $p_{m-j} \in S_{m_j}(\mathfrak{g}_a^* M)$, $j = 1, 2$, then the computation of $p_{m_1} * p_{m_2}$ at a point $\xi \in \mathfrak{g}^* M \setminus 0$ requires the knowledge of the values p_{m_1} and p_{m_2} at all the points ξ' of $\mathfrak{g}_a^* M$. It follows that the product (5) for Heisenberg symbols is not microlocal, i.e., it cannot be localized with respect to the ξ -variable.

As we will explain in Section 4 this lack of microlocality prevents us from carrying out Seeley's approach to complex powers. Nevertheless, complex powers of hypoelliptic Ψ DO's were dealt with in [Po2] by relying on a pseudodifferential representation of the heat kernel, instead of using a pseudodifferential representation of the resolvent as in Seeley's approach. In particular, assuming M and \mathcal{E} endowed with a Riemannian metric and a Hermitian metric, we have:

Proposition 2.1 ([Po2]). *Let $P : C^\infty(M, \mathcal{E}) \rightarrow C^\infty(M, \mathcal{E})$ be a differential operator ≥ 0 of Heisenberg order m such that $\sigma_m(P)$ is invertible. Then, for any $s \in \mathbb{C}$, the power P^s defined by L^2 -functional calculus is a Ψ_H DO of order ms .*

3. COMPLEX POWERS OF THE CONTACT LAPLACIAN

Let M^{2n-1} be a compact orientable contact manifold with contact hyperplane $H = \ker \theta$, where θ is a contact form and let J be an almost complex structure on H such that we have $d\theta(X, JX) > 0$ for any section X of $H \setminus 0$. We then can endow M with the Riemannian metric $g_{\theta, J} = \theta^2 + d\theta(\cdot, J\cdot)$. In addition, we let X_0 be the Reeb vector field associated to θ , so that $\theta(X_0) = 1$ and $\iota_{X_0} d\theta = 0$.

The splitting $TM = \mathbb{R}X_0 \oplus H$ allows us to identify H^* with the annihilator of X_0 in T^*M and to identify $\Lambda_{\mathbb{C}}^k H^*$ with $\ker \iota_{X_0}$, so that we get the orthogonal splitting $\Lambda_{\mathbb{C}}^* TM = (\bigoplus_{k=0}^{2n} \theta \wedge \Lambda_{\mathbb{C}}^k H^*) \oplus (\bigoplus_{k=0}^{2n} \Lambda_{\mathbb{C}}^k H^*)$. If $\eta \in C^\infty(M, \Lambda_{\mathbb{C}}^k H^*)$ then we have $d\eta = \theta \wedge \mathcal{L}_{X_0} \eta + d_b \eta$, where $d_b \eta$ is the component of $d\eta$ in $\Lambda_{\mathbb{C}}^k H^*$. This does not provide us with a complex, for we have $d_b^2 = -\mathcal{L}_{X_0} \varepsilon(d\theta)$, where $\varepsilon(d\theta)$ denotes the exterior multiplication by $d\theta$.

The contact complex of Rumin [Ru] arises as a modification d_b and $\Lambda_{\mathbb{C}}^k H^*$ to get a complex of horizontal differential forms. Let $\Lambda_1^* := \ker \iota(d\theta) \cap \Lambda_{\mathbb{C}}^* H^*$ and $\Lambda_2^* := \ker \varepsilon(d\theta) \cap \Lambda_{\mathbb{C}}^* H^*$. Then $C^\infty(M, \Lambda_2^*)$ is closed under d_b and annihilates d_b^2 and $C^\infty(M, \Lambda_2^*)$ is closed under d_b^* and annihilated by $(d_b^*)^2$, so that we get two complexes. However, since $d\theta$ is nondegenerate on H the operator $\varepsilon(d\theta) : \Lambda_{\mathbb{C}}^k H^* \rightarrow \Lambda_{\mathbb{C}}^{k+2} H^*$ is injective for $k \leq n-2$ and surjective for $k \geq n$ and so $\Lambda_2^* = \{0\}$ for $k \leq n-1$ and $\Lambda_1^* = \{0\}$ for $k \geq n$. Therefore, we only have two halves of complexes.

As observed by Rumin [Ru] we get a full complex by connecting the two halves by means of the differential operator $D_{R; n-1} := \mathcal{L}_{X_0} + d_{b; n-2} \varepsilon(d\theta)^{-1} d_{b; n-1}$ acting on $C^\infty(M, \Lambda_{\mathbb{C}}^{n-1} H^*)$, where $\varepsilon(d\theta)^{-1}$ is the inverse of $\varepsilon(d\theta) : \Lambda_{\mathbb{C}}^{n-2} H^* \rightarrow \Lambda_{\mathbb{C}}^n H^*$. Therefore, if we let $\pi_1 \in C^\infty(M, \Lambda_{\mathbb{C}}^* H^*)$ be the orthogonal projection onto Λ_1 then we have the complex,

$$(7) \quad C^\infty(M) \xrightarrow{d_{R; 0}} \dots C^\infty(M, \Lambda^{n-2}) \xrightarrow{D_{R; n-1}} C^\infty(M, \Lambda^{n-1}) \dots \xrightarrow{d_{R; 2n-3}} C^\infty(M, \Lambda^{2n-2}).$$

where $d_{R; k}$ agrees with $\pi_1 \circ d_b$ for $k = 0, \dots, n-2$ and with $d_{R; k} = d_b$ otherwise.

The contact Laplacian is defined as follows. In degree $k \neq n$ this is the differential operator $\Delta_{R;k} : C^\infty(M, \Lambda^k) \rightarrow C^\infty(M, \Lambda^k)$ such that

$$(8) \quad \Delta_{R;k} = \begin{cases} (n-1-k)d_{R;k-1}d_{R;k}^* + (n-k)d_{R;k+1}^*d_{R;k}, & k = 0, \dots, n-2, \\ (k-n)d_{R;k-1}d_{R;k}^* + (k-n+1)d_{R;k+1}^*d_{R;k}, & k = n, \dots, 2n. \end{cases}$$

For $k = n-1$ we have the differential operators $\Delta_{R;n-1,j} : C^\infty(M, \Lambda_j^n) \rightarrow C^\infty(M, \Lambda_j^n)$, $j = 1, 2$, given by the formulas,

$$(9) \quad \Delta_{R;n-1,1} = (d_{R;n-2}d_{R;n-1}^*)^2 + D_{R;n-1}^*D_{R;n-1},$$

$$(10) \quad \Delta_{R;n-1,2} = D_{R;n-1}D_{R;n-1}^* + (d_{R;n}^*d_{R;n-1}).$$

Notice that $\Delta_{R;k}$, $k \neq n-1$, is a differential operator of (Heisenberg) order 2, while $\Delta_{R;n-1,j}$, $j = 1, 2$, is a differential operator of (Heisenberg) order 4.

It has been shown by Rumin [Ru] that the contact Laplacian is hypoelliptic. In fact, we have:

Proposition 3.1 ([JK], [Po2]). *The principal symbols $\sigma_2(\Delta_{R;k})$, $k \neq n-1$, and $\sigma_4(\Delta_{R;n-1,j})$, $j = 1, 2$, are invertible with the respect to the product (5) for Heisenberg symbols.*

Combining this with Proposition 2.1 then gives:

Proposition 3.2 ([Po2]). *For any $s \in \mathbb{C}$ the powers $\Delta_{R;k}^s$, $k \neq n-1$, and $\Delta_{R;n-1,j}^s$, $j = 1, 2$, defined by L^2 -functional calculus are $\Psi_H DO$'s of degree $2s$ and $4s$ respectively.*

This fills the gap in [JK] alluded to in Section 1.

4. HEISENBERG CALCULUS AND SEELEY'S APPROACH

In this last section we explain why the lack of microlocality of the Heisenberg calculus actually prevents us from implementing into this setting Seeley's approach to complex powers.

4.1. Seeley's approach to complex powers. Let us briefly recall the approach of Seeley [Se] to complex powers (see also [GS], [Sh]). To simplify the exposition we let M^n be a compact Riemannian manifold equipped and let $\Delta : C^\infty(M) \rightarrow C^\infty(M)$ be a second order positive elliptic differential operator with principal symbol $p_2(x, \xi) > 0$. Then for $\Re s < 0$ we have:

$$(11) \quad \Delta^s = \frac{i}{2\pi} \int_{\Gamma_r} \lambda^s (\Delta - \lambda)^{-1} d\lambda,$$

$$(12) \quad \Gamma_r = \{\rho e^{i\pi}; \infty < \rho \leq r\} \cup \{r e^{it}; \theta \geq t \geq \theta - 2\pi\} \cup \{\rho e^{-i\pi}; r \leq \rho \leq \infty\},$$

where $r > 0$ is small enough so that non nonzero eigenvalue of Δ lies in $(0, r]$.

To show that the formula above defines a ΨDO Seeley constructs an asymptotic resolvent $Q(\lambda)$ as a parametrix for $\Delta - \lambda$ in a suitable ΨDO calculus with parameter. More precisely, let $\Lambda \subset \mathbb{C} \setminus 0$ be an open angular sector $\theta < \arg \lambda < \theta'$ with $0 < \theta < \pi < \theta' < 2\pi$. In the sequel we will say that a subset $\Theta \subset [\mathbb{R}^n \times \mathbb{C}] \setminus 0$ is *conic* when for any $t > 0$ and any $(\xi, \lambda) \in \Theta$ we have $(t\xi, t^2\lambda) \in \Theta$. For instance the subset $\mathbb{R}^n \times \Lambda \subset [\mathbb{R}^n \times \mathbb{C}] \setminus 0$ is conic.

Let $U \subset \mathbb{R}^n$ be a local chart for M . Then in U the asymptotic resolvent has a symbol of the form $q(x, \xi; \lambda) \sim \sum_{j \geq 0} q_{-2-j}(x, \xi; \lambda)$, where \sim is taken in a suitable

sense (see [Se]) and there exists an open conic subset $\Theta \subset [\mathbb{R}^n \times \mathbb{C}] \setminus 0$ containing $\mathbb{R}^n \times \Lambda$ such that each symbol $q_{-2-j}(x, \xi; \lambda)$ is smooth on $U \times \Theta$ and satisfies

$$(13) \quad q_{-2-j}(x, t\xi; t^2\lambda) = t^{-2-j}q(x, \xi; \lambda) \quad \forall t > 0.$$

If $p(x, \xi) = \sum_{j=0}^2 p_{2-j}(x, \xi)$ denotes the symbol of Δ in the local chart U then $q(x, \xi; \lambda)$ is such that $1 \sim (p(x, \xi) - \lambda)q(x, \xi; \lambda) + \sum_{\alpha \neq 0} \frac{1}{\alpha!} \partial_\xi^\alpha p(x, \xi) D_x^\alpha q(x, \xi; \lambda)$, from which we get

$$(14) \quad q_{-2}(x, \xi; \lambda) = (p_2(x, \xi) - \lambda)^{-1},$$

$$(15) \quad q_{-2-j}(x, \xi; \lambda) = -q_{-2}(x, \xi; \lambda) \sum_{\substack{|\alpha|+k+l=j, \\ l \neq j}} \frac{1}{\alpha!} \partial_\xi^\alpha p_{2-k}(x, \xi) D_x^\alpha q_{-2-l}(x, \xi; \lambda).$$

Set $\rho = \inf_{x \in U} \inf_{|\xi|=1} p_2(x, \xi)$. Possibly by shrinking U we may assume $\rho > 0$. Let $\Theta = [\mathbb{R}^n \times \Lambda] \cup \{(\xi; \lambda) \in \mathbb{R}^n \times \mathbb{C}; 0 \leq |\lambda| < \rho|\xi|^2\}$. Then the formulas (14) and (15) show that each symbol $q_{-2-j}(x, \xi; \lambda)$ is well defined and smooth on $U \times \Theta$ and is homogeneous in the sense of (13). Furthermore, it is analytic with respect to λ . Therefore, for $\Re s < 0$ we define a smooth function on $U \times (\mathbb{R}^n \setminus 0)$ by letting

$$(16) \quad p_{s, ms-j}(x, \xi) = \frac{i}{2\pi} \int_{\Gamma_\xi} \lambda^s q_{-2-j}(x, \xi; \lambda) d\lambda,$$

where Γ_ξ is the contour Γ_r in (12) with $r = \frac{1}{2}\rho|\xi|^2$. Moreover, one can check that $p_{s, ms-j}(x, t\xi) = t^{2s-j}p_{s, ms-j}(x, \xi)$ for any $t > 0$, i.e., $p_{s, ms-j}(x, \xi)$ is a homogeneous symbol of degree $ms - j$.

It can also be shown that on the chart U the operators Δ^s is a Ψ DO with symbol $p_s \sim \sum_{j \geq 0} p_{s, ms-j}(x, \xi)$. This is true on any local chart and we can check that the Schwartz kernel of Δ^s is smooth off the diagonal of $M \times M$, so we see that Δ^s is a Ψ DO of order ms for $\Re s < 0$.

Finally, for $\Re s \geq 0$ and k integer $> \Re s$ we have $\Delta^s = \Delta^k \Delta^{s-k}$, so since Δ^k is a differential operator of order k and Δ^{s-k} is a Ψ DO of order $m(s-k)$ we see that Δ^s is a Ψ DO of order ms . Hence Δ^s is a Ψ DO of order ms for any $s \in \mathbb{C}$.

4.2. Obstruction to Seeley's approach. Let us now explain why we cannot carry out Seeley's approach within the framework of the Heisenberg calculus. We will explain this in the special case of the contact Laplacian $\Delta_{R;0}$ acting on the functions on a compact orientable contact manifold (M^{2n-1}, H) as in Section 3.

In order to carry out Seeley's approach for $\Delta_{R;0}$ we have to construct an asymptotic resolvent in a class of Ψ_H DO's with parameter associated to an angular sector $\Lambda \subset \mathbb{C} \setminus [0, \infty)$ as above and given in a local Heisenberg chart $U \subset \mathbb{R}^{2n-1}$ by parametric symbols, $q(x, \xi; \lambda) \sim \sum_{j \geq 0} q_{-2-j}(x, \xi; \lambda)$, where \sim is taken in a suitable sense and there exists an open conic subset $\Theta \subset [\mathbb{R}^n \times \mathbb{C}] \setminus 0$ containing $\mathbb{R}^{2n-1} \times \Lambda$ such that each symbol $q_{-2-j}(x, \xi; \lambda)$ is smooth on $U \times \Theta$ and satisfies $q_{-2-j}(x, t\xi; t^2\lambda) = t^{-2-j}q(x, \xi; \lambda)$ for any $t > 0$.

If we let $p(x, \xi) = \sum p_{2-j}(x, \xi)$ be the symbol of $\Delta_{R;0}$ in the Heisenberg chart, then by (6) we have

$$(17) \quad 1 \sim \sum_{j \geq 0} \sum_{k+l \leq j} \sum_{\alpha, \beta, \gamma, \delta}^{(j-k-l)} h_{\alpha\beta\gamma\delta}(x) (D_\xi^\delta p_{2-k}) * (\xi^\gamma \partial_x^\alpha \partial_\xi^\beta q_{-2-l})(x, \xi; \lambda),$$

from which we get

$$(18) \quad q_{-2}(x, \xi; \lambda) = (p_2 - \lambda)^{* -1}(x, \xi; \lambda)$$

$$(19) \quad$$

$$q_{-2-j}(x, \xi; \lambda) = - \sum_{\substack{k+l \leq j \\ l \neq j}} \sum_{\alpha, \beta, \gamma, \delta}^{(j-k-l)} h_{\alpha\beta\gamma\delta}(x) q_{-2} * (D_\xi^\delta p_{2-k}) * (\xi^\gamma \partial_x^\alpha \partial_\xi^\beta q_{-2-l})(x, \xi; \lambda),$$

where $(p_2 - \lambda)^{* -1}$ denotes the inverse of $p_2 - \lambda$ with respect to the product $*$.

If $q_1(x, \xi; \lambda)$ and $q_2(x, \xi; \lambda)$ are two homogeneous Heisenberg symbols with parameter then the product q_1 and q_2 should be defined as

$$(20) \quad q_1 * q_2(x, \xi; \lambda) = [q_1(x, \cdot; \lambda) *^x q_2(x, \cdot; \lambda)](\xi).$$

As mentioned in Section 2 the definition of $[q_1(x, \cdot; \lambda) *^x q_2(x, \cdot; \lambda)](\xi)$ depends on all the values of $q_1(x, \xi'; \lambda)$ and $q_2(x, \xi'; \lambda)$ as ξ' ranges over $\mathbb{R}^{2n-1} \setminus 0$. For a parameter $\lambda > 0$ the symbols $q_1(x, \xi; \lambda)$ and $q_2(x, \xi; \lambda)$ are only defined for ξ in $\{\xi; (x, \xi; \lambda) \in \Theta\}$ which does not agree with $\mathbb{R}^{2n-1} \setminus 0$, so we cannot define $q_1 * q_2(x, \xi; \lambda)$ for $\lambda > 0$. Therefore, the formula (19) does not make sense for $\lambda > 0$.

All this shows that the non-microlocality of the Heisenberg calculus prevents us from implementing Seeley's approach into the setting of the Heisenberg calculus. As previously mentioned the results of [Po2] allows us to deal with complex powers in case of positive differential operators with invertible principal symbols. It also possible to construct complex powers for more general hypoelliptic Ψ DO's in the spirit of Seeley's approach by replacing the use of a homogeneous symbols with parameter by *almost* homogeneous symbols with parameter (see [Po1], [Po3]).

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